

Periodic Solution of a Certain Non-Linear Third Order Differential Equation

B. Mehri* and D. Shadman¹

In this paper, the third order differential equation $x''' + \psi(x')x'' + (k^2 + \phi(x))x' + f(t, x) = \varepsilon(t)$ is considered. Under certain conditions on the functions appearing in the differential equation, the existence of periodic solutions is proved. Similar problems have been treated by authors in [4,6,7]. However, the method employed here is used by Reissig [1] and the results obtained are, in fact, a generalization of those in [1-3]. The conditions imposed on the nonlinear terms do not require the ultimate boundedness of all solutions.

INTRODUCTION

Frequently studies in physical, biological and social sciences require information about the existence or nonexistence of periodic behavior of the related systems. In many cases the mathematical models obtained for such systems are nonlinear ordinary differential equations.

Extensive studies have been done on the existence of periodic solutions of second order differential equations. Here, a certain third order nonlinear differential equation is treated, where the existence of the periodic solutions is proved using the bounded results discussed in Theorems 1 and 2

Here, the following differential equation is considered:

$$x''' + \psi(x')x'' + (k^2 + \phi(x))x' + f(t, x) = \varepsilon(t), \quad (1)$$

where the function $\psi(x')$, is continuous, $f(t, x)$ and $\varepsilon(t)$ are continuous and periodic in t of period ω and $\phi(x)$ is such that $\Phi(x) = \int_0^x \phi(t)dt$ is continuous.

THEOREM 1

Differential Equation 1 admits at least one ω -periodic solution, if:

i) $0 < k \leq \frac{\pi}{\omega}$,

*. Corresponding Author, Department of Mathematical Sciences, Sharif University of Technology, Tehran, I.R. Iran.

1. Department of Mathematical Sciences, Sharif University of Technology, Tehran, I.R. Iran.

- ii) $\int_0^\omega \varepsilon(t)dt = 0$ that is $E(t) = \int_0^t \varepsilon(\tau)d\tau$ is ω -periodic,
- iii) $|\Psi(y)| \leq M$, $\Psi(y) = \int_0^y \psi(\tau)d\tau$,
- iv) $\frac{|\Phi(x)|}{|x|} \rightarrow 0$ ($|x| \rightarrow \infty$), $\Phi(x) = \int_0^x \phi(\tau)d\tau$,
- v) $\frac{|f(t,x)|}{|x|} \rightarrow 0$ $|x| \rightarrow \infty$, uniformly in t ,
- vi) $f(t, x) \text{ sign } x \geq 0, |x| \geq h$.

Proof

The proof, described by means of the Leray-Schauder method, is simple. A differential equation containing the parameter $\mu, 0 \leq \mu \leq 1$ is described.

$$x''' + k^2x' + cx = \mu\{\varepsilon(t) - f(t, x) + cx - \frac{d}{dt}[\Psi(x') + \Phi(x)]\}, \quad (2)$$

where c is an arbitrary positive constant. For $\mu = 0$, a homogeneous linear equation is obtained, the only ω -periodic solution of which is the trivial one. For $\mu = 1$, Equation 2 is identical to Equation 1. It is a well-known fact [2,5] that Equation 2 admits at least one periodic solution for each parameter value $\mu \in [0, 1]$ if, for $0 < \mu < 1$, all periodic solutions, as well as their derivatives of the first and second order, are uniformly bounded. Consequently, the stated theorem can be proved with the aid of an a priori bound.

Let $x(t) \equiv x(t + \omega)$ be a solution of Equation 2

and let $0 < \mu < 1$. It is denoted that:

$$R = \max_{0 \leq t \leq \omega} |x(t)|,$$

$$F = F(R) = \max_{0 \leq t \leq \omega, |x| \leq R} |f(t, x)|,$$

$$\Phi(R) = \max_{|x| \leq R} |\Phi(x)|.$$

The derivative $y = x'$ fulfills the equation:

$$y'' + k^2 y = \mu \{p(t) - f(t, x) - \frac{d}{dt} [\Psi(y) + \Phi(x)]\} - (1 - \mu)cx(t),$$

which can be written as a non-homogeneous linear equation:

$$y'' + k^2 y = q(t), \quad q(t + \omega) = q(t), \tag{3}$$

where:

$$q(t) = \mu \{ \varepsilon(t) - f(t, x(t)) - \frac{d}{dt} [\Psi(y(t)) + \Phi(x(t))] \} - (1 - \mu)cx(t). \tag{4}$$

Now, let $G(t, s)$ be Green's function for the following equation:

$$y'' + k^2 y = 0,$$

with the periodic boundary conditions of $y^{(i)}(0) = y^{(i)}(\omega)$ and $i = 0, 1$ of the following form:

$$G(t, s) = \frac{1}{2k^2 \sin \frac{k\omega}{2}} \begin{cases} \cos k(\frac{\omega}{2} + s - t); & 0 \leq s \leq t \leq \omega \\ \cos k(\frac{\omega}{2} + t - s); & 0 \leq t \leq s \leq \omega \end{cases}$$

Obviously, $\frac{\partial^2 G}{\partial t^2} = (-1)^i \frac{\partial^2 G}{\partial s^2}, i = 0, 1, |G(t, s)| \leq \frac{\pi}{2\omega k^3}, |\frac{\partial G}{\partial t}| \leq \frac{\pi}{2\omega k^2}$ and $|\frac{\partial^2 G}{\partial t^2}| \leq \frac{\pi}{2\omega k}$. Using Green's function, the following representation is obtained for the solution $y(t)$ and $y'(t)$:

$$y(t) = \int_0^\omega G(t, s)q(s)ds, y'(t) = \int_0^\omega \frac{\partial G}{\partial t}(t, s)q(s)ds. \tag{5}$$

Next, $q(t)$ from Equation 4 is replaced into Expression 5, obtained for $y(t)$ and $y'(t)$. First, it is noted that:

$$\begin{aligned} & \int_0^\omega G(t, s) \frac{d}{ds} [\Psi(y(s)) + \Phi(x(s))] ds \\ &= G(t, s) [\Psi(y(s)) + \Phi(x(s))] \Big|_0^\omega \\ &- \int_0^\omega \frac{\partial G}{\partial s} [\Psi(y(s)) + \Phi(x(s))] ds \\ &= - \int_0^\omega \frac{\partial G}{\partial s}(t, s) [\Psi(y(s)) + \Phi(x(s))] ds, \end{aligned}$$

and:

$$\begin{aligned} & \int_0^\omega \frac{\partial G}{\partial t}(t, s) \frac{d}{ds} [\Psi(y(s)) + \Phi(x(s))] ds \\ &= - \int_0^\omega \frac{\partial G}{\partial s}(t, s) \frac{d}{ds} [\Psi(y(s)) + \Phi(x(s))] ds \\ &= -\Psi(y(t)) - \Phi(x(t)) - k^2 \int_0^\omega G(t, s) \\ & \quad [\Psi(y(s)) + \Phi(x(s))] ds. \end{aligned}$$

Using the above expressions, one obtains from Equations 4 and 5:

$$\begin{aligned} y(t) &= \int_0^\omega G(t, s) \{ \mu [\varepsilon(s) - f(s, x(s))] - (1 - \mu)cx(s) \} ds \\ &- \mu \int_0^\omega \frac{\partial G}{\partial s}(t, s) [\Psi(y(s)) + \Phi(x(s))] ds, \\ y'(t) &= \int_0^\omega \{ \mu [\varepsilon(s) - f(s, x(s))] - (1 - \mu)cx(s) \} ds \\ &- [\Psi(y(t)) + \Phi(x(t)) - k^2 \int_0^\omega G(t, s) [\Psi(y(s)) \\ & \quad + \Phi(x(s))] ds]. \end{aligned}$$

Now let $m = \int_0^\omega |\varepsilon(s)| ds$, then, one has:

$$\begin{aligned} |y(t)| &\leq (m + F(R) + cR) \int_0^\omega |G(t, s)| ds \\ &+ (M + \Phi(R)) \int_0^\omega \left| \frac{\partial G}{\partial s} \right| ds, \end{aligned}$$

and:

$$\begin{aligned} |y'(t)| &\leq (m + F(R) + cR) \int_0^\omega |G(t, s)| ds \\ &+ (M + \Phi(R)) + k^2 (M + \Phi(R)) \int_0^\omega |G(t, s)| ds, \end{aligned}$$

or:

$$\begin{aligned} |y(t)| &\leq (m + F(R) + cR) \frac{\pi}{2k^3} + (M + \Phi(R)) \frac{\pi}{2k^2}, \\ |y'(t)| &\leq (m + F(R) + cR) \frac{\pi}{2k^3} + (M + \Phi(R)) \\ & \quad + (M + \Phi(R)) \frac{\pi}{2k^2}. \end{aligned}$$

If one sets $\rho = \max \{ 1 + \frac{\pi}{2k}, \frac{\pi}{2k^2}, \frac{\pi}{2k^3} \}$, one obtains:

$$\begin{aligned} |y(t)| &\leq \rho(m + F(R) + cR + M + \Phi(R)), \\ |y'(t)| &\leq \rho(m + F(R) + cR + M + \Phi(R)). \end{aligned}$$

Assuming, $x(t)$ is a periodic solution, by substituting it in Equation 2 and integrating term by term, one obtains:

$$[x''(t) + k^2x(t) + [\Psi(x'(t)) + \Phi(x(t))]\mu - \mu E(t)]_0^\omega + \int_0^\omega \{(1 - \mu)cx(t) + \mu f(t, x(t))\}dt = 0.$$

Therefore:

$$\int_0^\omega \{(1 - \mu)cx(t) + f(t, x(t))\}dt = 0.$$

Since $(1 - \mu) > 0$,

$$(\text{sign } x)\{(1 - \mu)cx(t) + f(t, x(t))\} = (1 - \mu)c|x| + (\text{sign } x)f(t, x(t)) > 0, |x| > h.$$

Consequently, $|x(t)| \geq h$ for $0 \leq t \leq \omega$ is excluded. Therefore, there exists $0 < \tau < \omega$, such that $|x(\tau)| < h$. Now, using the mean-value theorem, one finds:

$$|x(t) - x(\tau)| = |t - \tau||y(\tau + \theta(t - \tau))| \leq \omega\rho(m + F(R) + cR + M + \Phi(R)),$$

or:

$$|x(t)| < h + \omega\rho(m + F(R) + cR + M + \Phi(R)),$$

which implies that:

$$R = \max_{0 \leq t \leq \omega} |x(t)| < h + \omega s(m + F(R) + cR + M + \Phi(R)).$$

Choosing $0 < c < \frac{1}{\omega\rho}$, one obtains:

$$1 < \frac{h + \omega\rho(m + M)}{1 - \omega c\rho} \frac{1}{R} + \frac{\omega\rho}{1 - \omega c\rho} \frac{F(R)}{R} + \frac{\omega.s}{1 - \omega.c.s} \frac{\Phi(R)}{R}. \tag{6}$$

An immediate consequence of (v) and (vi), is:

$$\frac{F(R)}{R} \rightarrow 0, \quad \frac{\Phi(R)}{R} \rightarrow 0, \quad R \rightarrow \infty.$$

Therefore, it is concluded from Equation 6 that:

$$R = \max_{0 \leq t \leq \omega} |x(t)| \leq R_0(\text{independent of } \mu),$$

$$F(R) = \max_{0 \leq t \leq \omega, |x| \leq R} |f(t, x)| \leq F_0 = \max_{0 \leq t \leq \omega, |x| \leq R_0} |f(t, x)|,$$

$$\Phi(R) = \max_{|x| \leq R} |\Phi(x)| \leq \Phi_0 = \max_{|x| \leq R_0} |\Phi(x)|.$$

The resulting a priori estimates:

$$|x(t)| \leq R_0, |x'(t)| \leq \rho m + F_0 + cR_0 + M + \Phi(R_0),$$

$$|x''(t)| \leq \rho(m + F_0 + CR_0 + M + \Phi(R_0)),$$

ensure the existence of a periodic solution of Equation 1 as stated in Theorem 1.

Remark

In the case:

$$\text{iv)}_2 f(t, x)\text{sign } x < 0, \quad |x| \geq h,$$

a new independent variable, $\tau = -t$, is introduced and a differential equation of the previous type is obtained. Thus, Theorem 1 remains valid if assumption vi) is replaced by vi)'.
 vi)' $f(t, x)\text{sign } x > 0, |x| \geq h$.

THEOREM 2

The differential equation:

$$x''' + (a + \psi(x'))x'' + \phi(x)x' + f(t, x) = \epsilon(t),$$

admits at least one ω -periodic solution if:

- i) $a \neq 0$,
- ii) $\int_0^\omega \epsilon(t)dt = 0$,
- iii) $|\Psi(y)| \leq M, \Psi(y) = \int_0^y \psi(s)ds$,
- iv) $\frac{|\Phi(x)|}{|x|} \rightarrow 0, |x| \rightarrow \infty, \Phi(x) = \int_0^x \phi(n)dn$,
- v) $\frac{|f(t, x)|}{|x|} \rightarrow 0(|x| \rightarrow \infty, \text{uniformly in } t), f(t + \omega, x) = f(t, x)$,
- vi)' $f(t, x)\text{sign } x > 0, |x| \geq h$.

Proof

Again, the proof is based on an a priori estimate of the ω -periodic solutions of the equation:

$$x'''(t) + ax''(t) + cx(t) = \mu\{\epsilon(t) - f(t, x)\} - \frac{d}{dt}[\Psi(x'(t)) + \Phi(x(t))] + cx(t). \tag{7}$$

Assuming $x(t) \equiv x(t + \omega)$ is a solution of Equation 7 and writing $x''(t) = z(t)$, one obtains:

$$z'(t) + az(t) = \mu\left\{\epsilon(t) - f(t, x(t)) - \frac{d}{dt}[\Psi(x'(t)) + \Phi(x(t))]\right\} - (1 - \mu)cx(t).$$

Now, let $G(t, s)$ be Green's function for the equation:

$$z' + az = 0,$$

with the boundary condition $z(0) = z(\omega)$ of the following form:

$$G(t, s) = \begin{cases} \frac{e^{a(s-t-\omega)}}{-1+e^{-a\omega}}, & 0 \leq t \leq s \leq \omega \\ \frac{e^{a(s-t)}}{-1+e^{-a\omega}}, & 0 \leq s \leq t \leq \omega \end{cases}$$

The following representation for solution $z(t)$ is obtained:

$$z(t) = \int_0^\omega G(t, s)q(s)ds,$$

where:

$$q(t) = \mu \left\{ c(t) - f(t, x(t)) - \frac{d}{dt} [\Psi(x'(t)) + \Phi(x(t))] \right\} - (1 - \mu)cx(t).$$

But:

$$\begin{aligned} & \int_0^\omega G(t, s) \frac{d}{ds} [\Psi(x'(s)) + \Phi(x(s))] ds \\ &= \int_0^{t-} G(t, s) \frac{d}{ds} [\Psi(x'(s)) + \Phi(x(s))] ds \\ & \quad + \int_{t+}^\omega G(t, s) \frac{d}{ds} [\Psi(x'(s)) + \Phi(x(s))] ds \\ &= \Psi(x'(s)) + \Phi(x(s)) \\ & \quad + a \int_0^\omega G(t, s) \frac{d}{ds} [\Psi(x'(s)) + \Phi(x(s))] ds. \end{aligned}$$

This implies that:

$$\begin{aligned} z(t) &= \int_0^\omega G(t, s) \{ \mu c(s) - f(s, x(s)) \\ & \quad - a[\Psi(x'(s)) + \Phi(x(s))] \\ & \quad - (1 - \mu)cx(s) \} ds \\ & \quad - \mu[\Psi(x'(s)) + \Phi(x(s))]. \end{aligned}$$

Assuming $\rho = 1 + a$, one obtains:

$$|z| \leq \rho(m + F(R) + M\Phi(R) + cR).$$

Now, since the trivial periodic solution $x(t)$ is excluded, one must have $\max y(t) > 0$ and $\min y(t) < 0$. Therefore, the application of the mean value theorem yields:

$$\max y(t) - \min y(t) \leq \omega \rho(m + f(R) + M + \Phi(R) + cR),$$

from which one concludes:

$$|y(t)| \leq \omega \rho(m + F(R) + M + \Phi(R) + cR).$$

Following the same argument in Theorem 1, similar boundedness results are obtained from which the existence of an ω -periodic solution is deduced.

REFERENCES

1. Reissig, R. "Periodic solutions of a third order nonlinear differential equation", *Ann. Mat. Pura Appl.*, **18**, pp 193-198 (1971).
2. Ezeilo, J.O.O. "A generalization of a theorem of Reissig for certain third order differential equation", *Ann. Mat. Pura. Appl.*, **87** pp 349-356 (1970).
3. Ezeilo, J.O.O. and Nkashama, M.N. "Resonant and nonresonant oscillations for some third order nonlinear ordinary differential equations", *Nonlinear Analysis Theory and Applications*, **12**(10), pp 1029-1046 (1988).
4. Mehri, B. and Shadman, D. "Boundedness of solutions of certain third order differential equation", *Math. Ineq. Appl.*, **2**(4), pp 545-549 (1999).
5. Shadman, D. and Mehri, B. "Periodic solutions of third order autonomous differential equations", *Proc. 3rd Seminar on Dynamical Systems*, Zanjan, Iran, pp 3-7 (1999).
6. Mehri, B. and Niksirat, M. "On the existence of periodic solutions for the quasi-linear third order differential equation", *J. Math. Analysis and Appl.*, **261**, pp 159-167 (2001).
7. Mehri, B. and Niksirat, M. "The existence of periodic solutions for the non-linear autonomous O.D.E'S.", *Nonlinear Analysis Forum*, **5**, pp 163-171 (2000).